Estimating parameters encoded in Gaussian transformations

RACQIT 2016
Universitat Autònoma de Barcelona, April 7^{th}, 2016

Nicolai Friis

Institute for Theoretical Physics, University of Innsbruck

in collaboration with

Michalis Skotiniotis, *Ivette Fuentes*, and *Wolfgang Dür*

Motivation & Outline

Motivation

- Quantum metrology: high-precision parameter estimation

  quantum strategies ⇒ scaling gap in resources
  w.r.t. best classical strategy ⇒ Heisenberg scaling
- Explore this advantage in relativistic scenarios?
- Typically described by Bogoliubov transformations

Outline

- Phase estimation: a paradigm metrology scenario
- General setup & the quantum Fisher information (QFI)
- Results [1]: pure states, tracing losses & optimal scaling

[1] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015) [arXiv:1502.07654].

Motivation & Outline

Motivation

- Quantum metrology: high-precision parameter estimation

  quantum strategies ⇒ scaling gap in resources
  w.r.t. best classical strategy ⇒ Heisenberg scaling

- Explore this advantage in relativistic scenarios?
- Typically described by Bogoliubov transformations

Outline

- Phase estimation: a paradigm metrology scenario
- General setup & the quantum Fisher information (QFI)
- Results [1]: pure states, tracing losses & optimal scaling

[1] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015) [arXiv:1502.07654].

Motivation & Outline

Motivation

- Quantum metrology: high-precision parameter estimation

  quantum strategies ⇒ scaling gap in resources
  w.r.t. best classical strategy ⇒ Heisenberg scaling
- Explore this advantage in relativistic scenarios?
- Typically described by Bogoliubov transformations

Outline

- Phase estimation: a paradigm metrology scenario
- General setup & the quantum Fisher information (QFI)
- Results [1]: pure states, tracing losses & optimal scaling

[1] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015) [arXiv:1502.07654].

Motivation & Outline

Motivation

- Quantum metrology: high-precision parameter estimation

  quantum strategies ⇒ scaling gap in resources
  w.r.t. best classical strategy ⇒ Heisenberg scaling
- Explore this advantage in relativistic scenarios?
- Typically described by Bogoliubov transformations

Outline

- Phase estimation: a paradigm metrology scenario
- General setup & the quantum Fisher information (QFI)
- Results [1]: pure states, tracing losses & optimal scaling

[1] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015) [arXiv:1502.07654].

Motivation & Outline

Motivation

- Quantum metrology: high-precision parameter estimation

  quantum strategies ⇒ scaling gap in resources
  w.r.t. best classical strategy ⇒ Heisenberg scaling
- Explore this advantage in relativistic scenarios?
- Typically described by Bogoliubov transformations

Outline

- Phase estimation: a paradigm metrology scenario
- General setup & the quantum Fisher information (QFI)
- Results [1]: pure states, tracing losses & optimal scaling

[1] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015) [arXiv:1502.07654].

Motivation & Outline

Motivation

- Quantum metrology: high-precision parameter estimation

  quantum strategies ⇒ scaling gap in resources
  w.r.t. best classical strategy ⇒ Heisenberg scaling
- Explore this advantage in relativistic scenarios?
- Typically described by Bogoliubov transformations

Outline

- Phase estimation: a paradigm metrology scenario
- General setup & the quantum Fisher information (QFI)
- Results [1]: pure states, tracing losses & optimal scaling

[1] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015) [arXiv:1502.07654].

Motivation & Outline

Motivation

- Quantum metrology: high-precision parameter estimation

  quantum strategies ⇒ scaling gap in resources
  w.r.t. best classical strategy ⇒ Heisenberg scaling

- Explore this advantage in relativistic scenarios?
- Typically described by Bogoliubov transformations

Outline

- Phase estimation: a paradigm metrology scenario
- General setup & the quantum Fisher information (QFI)
- Results [1]: pure states, tracing losses & optimal scaling

[1] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015) [arXiv:1502.07654].

Motivation & Outline

Motivation

- Quantum metrology: high-precision parameter estimation

  quantum strategies ⇒ scaling gap in resources
  w.r.t. best classical strategy ⇒ Heisenberg scaling
- Explore this advantage in relativistic scenarios?
- Typically described by Bogoliubov transformations

Outline

- Phase estimation: a paradigm metrology scenario
- General setup & the quantum Fisher information (QFI)
- Results [1]: pure states, tracing losses & optimal scaling

[1] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015) [arXiv:1502.07654].

Motivation & Outline

Motivation

- Quantum metrology: high-precision parameter estimation

  quantum strategies ⇒ scaling gap in resources
  w.r.t. best classical strategy ⇒ Heisenberg scaling
- Explore this advantage in relativistic scenarios?
- Typically described by Bogoliubov transformations

Outline

- Phase estimation: a paradigm metrology scenario
- General setup & the quantum Fisher information (QFI)
- Results [1]: pure states, tracing losses & optimal scaling

[1] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015) [arXiv:1502.07654].

Motivation & Outline

Motivation

- Quantum metrology: high-precision parameter estimation

  quantum strategies ⇒ scaling gap in resources
  w.r.t. best classical strategy ⇒ Heisenberg scaling

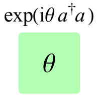
- Explore this advantage in relativistic scenarios?
- Typically described by Bogoliubov transformations

Outline

- Phase estimation: a paradigm metrology scenario
- General setup & the quantum Fisher information (QFI)
- Results [1]: pure states, tracing losses & optimal scaling

[1] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015) [arXiv:1502.07654].

Phase estimation

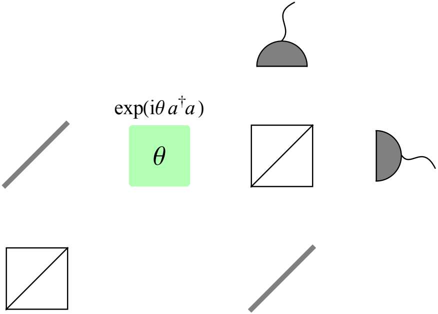


n single photon probes, count output photons $N_C = c^\dagger c$

repeat ν times: $\Delta N_C \rightarrow \Delta\bar{\theta} \propto 1/\sqrt{n\nu}$ "shot noise"

[2] V. Giovanetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004) [arXiv:quant-ph/0412078].

Phase estimation

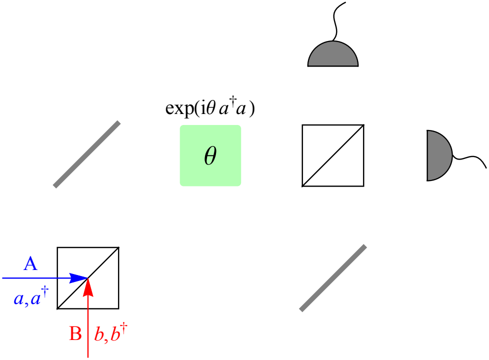


n single photon probes, count output photons $N_C = c^\dagger c$

repeat ν times: $\Delta N_C \rightarrow \Delta\bar{\theta} \propto 1/\sqrt{n\nu}$ "shot noise"

[2] V. Giovanetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004) [arXiv:quant-ph/0412078].

Phase estimation

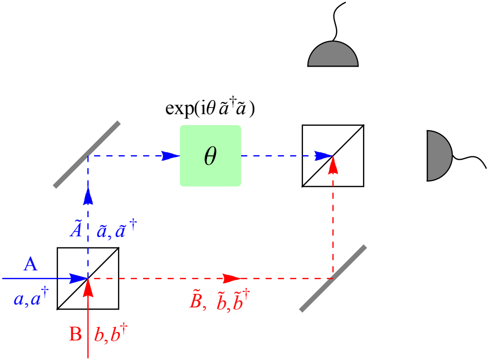


n single photon probes, count output photons $N_C = c^\dagger c$

repeat ν times: $\Delta N_C \rightarrow \Delta\bar{\theta} \propto 1/\sqrt{n\nu}$ "shot noise"

[2] V. Giovanetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004) [arXiv:quant-ph/0412078].

Phase estimation

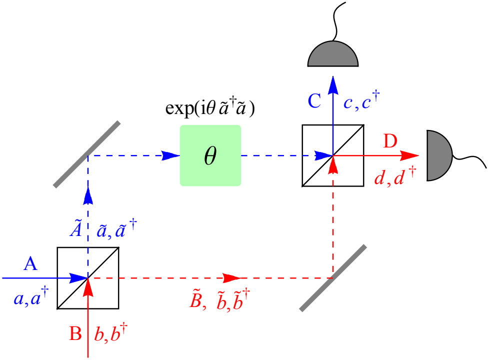


n single photon probes, count output photons $N_C = c^\dagger c$

repeat ν times: $\Delta N_C \rightarrow \Delta\bar{\theta} \propto 1/\sqrt{n\nu}$ "shot noise"

[2] V. Giovanetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004) [arXiv:quant-ph/0412078].

Phase estimation

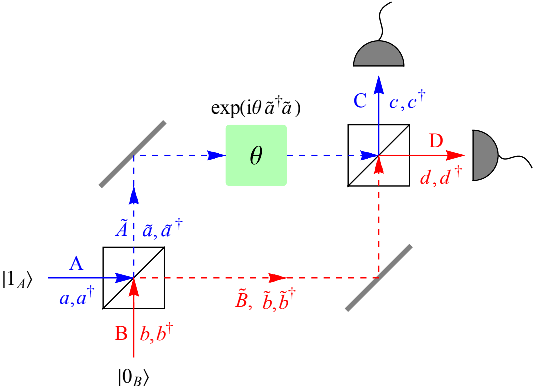


n single photon probes, count output photons $N_C = c^\dagger c$

repeat ν times: $\Delta N_C \rightarrow \Delta\bar{\theta} \propto 1/\sqrt{n\nu}$ "shot noise"

[2] V. Giovanetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004) [arXiv:quant-ph/0412078].

Phase estimation

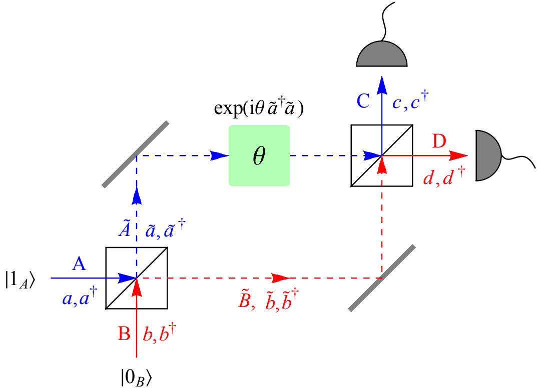


n single photon probes, count output photons $N_C = c^\dagger c$

repeat ν times: $\Delta N_C \to \Delta\bar{\theta} \propto 1/\sqrt{n\nu}$ "shot noise"

[2] V. Giovanetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004) [arXiv:quant-ph/0412078].

Phase estimation

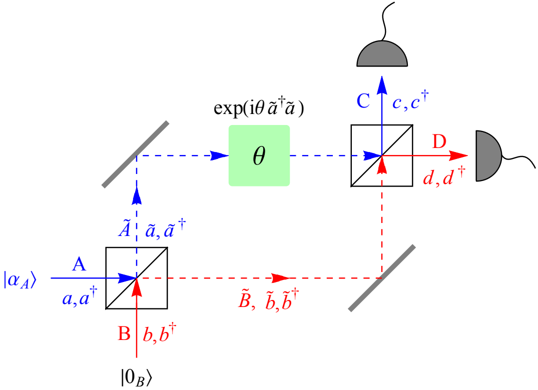


n single photon probes, count output photons $N_C = c^\dagger c$

repeat ν times: $\Delta N_C \rightarrow \Delta\bar{\theta} \propto 1/\sqrt{n\nu}$ "shot noise"

[2] V. Giovanetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004) [arXiv:quant-ph/0412078].

Phase estimation

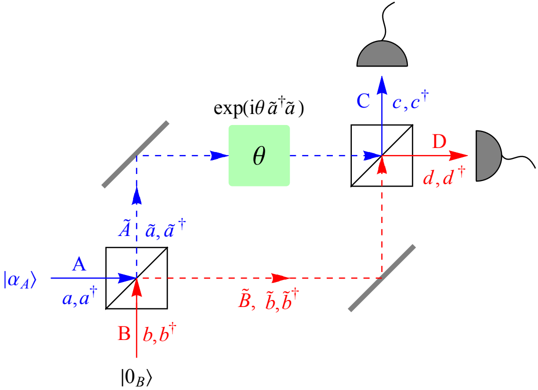


Alternatively: coherent state $|\alpha_A\rangle$ with $|\alpha|^2 = n$

n photons at once (ν times), still: $\Delta\bar{\theta} \propto 1/\sqrt{n\nu}$ "shot noise"

[2] V. Giovanetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004) [arXiv:quant-ph/0412078].

Phase estimation

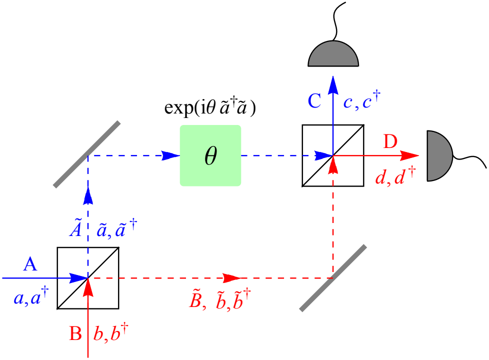


Alternatively: coherent state $|\alpha_A\rangle$ with $|\alpha|^2 = n$

n photons at once (ν times), still: $\Delta\bar{\theta} \propto 1/\sqrt{n\nu}$ "shot noise"

[2] V. Giovanetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004) [arXiv:quant-ph/0412078].

Phase estimation



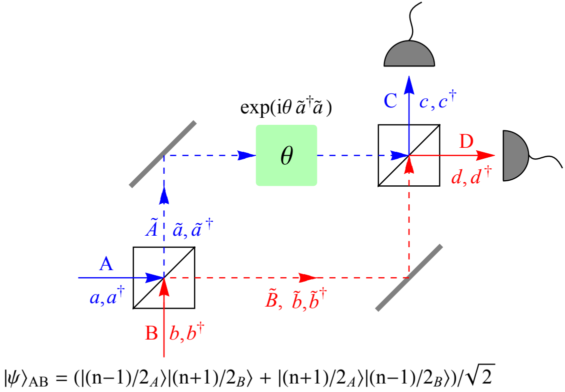
$$|\psi\rangle_{AB} = (|(n-1)/2_A\rangle|(n+1)/2_B\rangle + |(n+1)/2_A\rangle|(n-1)/2_B\rangle)/\sqrt{2}$$

Entangled input state $|\psi\rangle_{AB}$, average photon number n

quadratic improvement: $\Delta\bar{\theta} \propto 1/(n\sqrt{\nu})$ Heisenberg scaling

[2] V. Giovanetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004) [arXiv:quant-ph/0412078].

Phase estimation

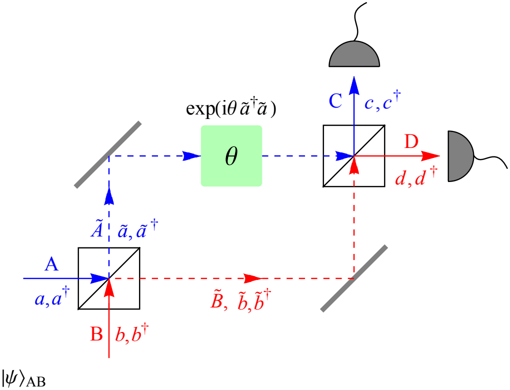


Entangled input state $|\psi\rangle_{AB}$, average photon number n

quadratic improvement: $\Delta\bar{\theta} \propto 1/(n\sqrt{\nu})$ Heisenberg scaling

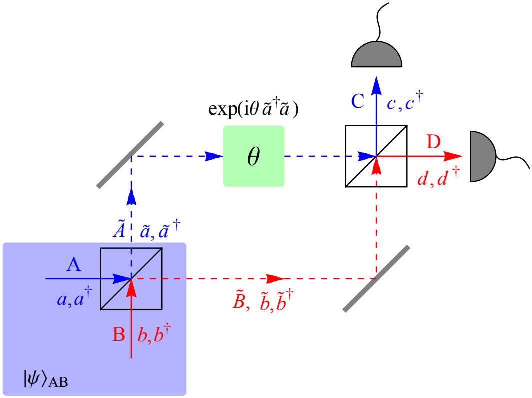
[2] V. Giovanetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004) [arXiv:quant-ph/0412078].

Generalized Setting



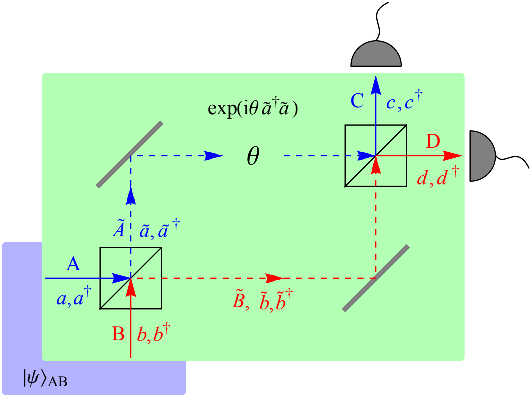
Stages: Preparation, Transformation, Measurement

Generalized Setting



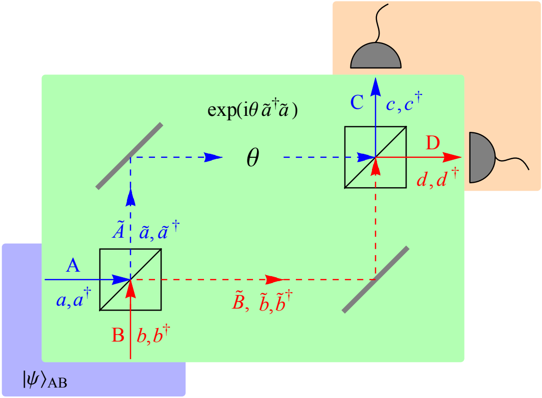
Stages: Preparation, Transformation, Measurement

Generalized Setting



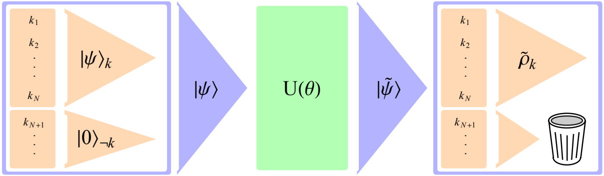
Stages: Preparation, Transformation, Measurement

Generalized Setting



Stages: Preparation, Transformation, Measurement

Generalized Setting



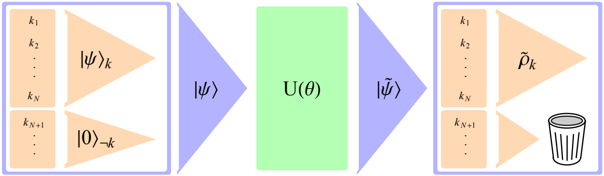
Preparation: only N modes controlled: $|\psi\rangle = |\psi\rangle_k \, |0\rangle_{\neg k}$

Transformation: (Gaussian) unitary on arbitrary many modes

Measurement: trace out $\neg k$, ν repetitions

[1] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015) [arXiv:1502.07654].

Generalized Setting



Preparation: only N modes controlled: $|\psi\rangle = |\psi\rangle_k\,|0\rangle_{\neg k}$

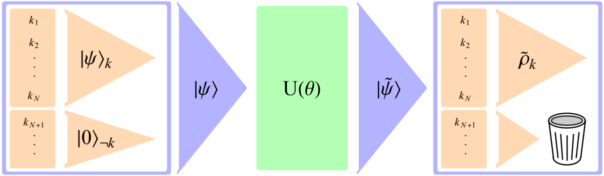
Transformation: (Gaussian) unitary on arbitrary many modes

Measurement: trace out $\neg k$, ν repetitions

Optimize over measurements $\Rightarrow$ Quantum Fisher Info $\mathcal{I}(\rho(\theta))$

[1] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015) [arXiv:1502.07654].

Generalized Setting



Preparation: only N modes controlled: $|\psi\rangle = |\psi\rangle_k \, |0\rangle_{\neg k}$

Transformation: (Gaussian) unitary on arbitrary many modes

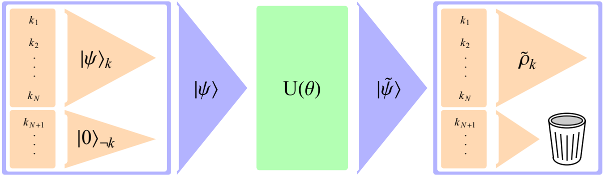
Measurement: trace out $\neg k$, ν repetitions

Optimize over measurements $\Rightarrow$ Quantum Fisher Info $\mathcal{I}(\rho(\theta))$

Cramér-Rao inequality $\Delta\bar{\theta} \geq 1/\sqrt{\nu \mathcal{I}(\rho(\theta))}$ with

[1] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015) [arXiv:1502.07654].

Generalized Setting



Preparation: only N modes controlled: $|\psi\rangle = |\psi\rangle_k\, |0\rangle_{\neg k}$

Transformation: (Gaussian) unitary on arbitrary many modes

Measurement: trace out $\neg k$, ν repetitions

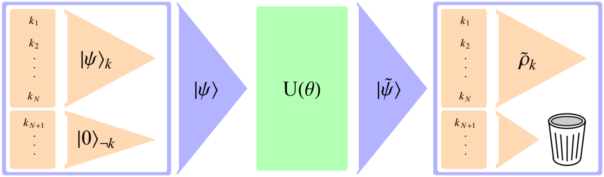
Optimize over measurements $\Rightarrow$ Quantum Fisher Info $\mathcal{I}(\rho(\theta))$

Cramér-Rao inequality $\Delta\bar{\theta} \geq 1/\sqrt{\nu\mathcal{I}(\rho(\theta))}$ with

$$\mathcal{I}(\rho(\theta)) = \lim_{d\theta\to 0} 8\frac{1-\sqrt{\mathcal{F}(\rho(\theta),\rho(\theta+d\theta))}}{d\theta^2}$$

[1] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015) [arXiv:1502.07654].

Generalized Setting



Preparation: only N modes controlled: $|\psi\rangle = |\psi\rangle_k |0\rangle_{\neg k}$

Transformation: (Gaussian) unitary on arbitrary many modes

Measurement: trace out $\neg k$, ν repetitions

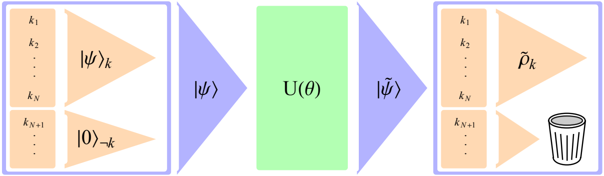
Optimize over measurements $\Rightarrow$ Quantum Fisher Info $\mathcal{I}(\rho(\theta))$

Cramér-Rao inequality $\Delta\bar{\theta} \geq 1/\sqrt{\nu\mathcal{I}(\rho(\theta))}$ with

$$\mathcal{I}(\rho(\theta)) = \lim_{d\theta\to 0} 8\frac{1-\sqrt{\mathcal{F}\big(\rho(\theta),\rho(\theta+d\theta)\big)}}{d\theta^2}$$

[1] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015) [arXiv:1502.07654].

Generalized Setting



Preparation: only N modes controlled: $|\psi\rangle = |\psi\rangle_k |0\rangle_{\neg k}$

Transformation: (Gaussian) unitary on arbitrary many modes

Measurement: trace out $\neg k$, ν repetitions

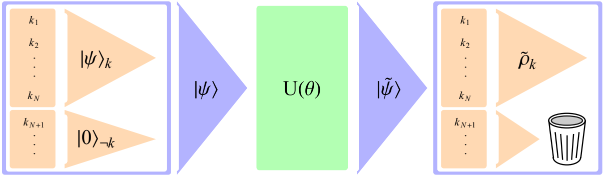
Optimize over measurements $\Rightarrow$ Quantum Fisher Info $\mathcal{I}(\rho(\theta))$

Cramér-Rao inequality $\Delta\bar{\theta} \geq 1/\sqrt{\nu \mathcal{I}(\rho(\theta))}$ with

$$\mathcal{I}\big(\rho(\theta)\big) = \lim_{d\theta \to 0} 8 \frac{1-\sqrt{\mathcal{F}\big(\rho(\theta),\rho(\theta+d\theta)\big)}}{d\theta^2}$$

[1] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015) [arXiv:1502.07654].

Generalized Setting



Preparation: only N modes controlled: $|\psi\rangle = |\psi\rangle_k \, |0\rangle_{\neg k}$

Transformation: (Gaussian) unitary on arbitrary many modes

Measurement: trace out $\neg k$, ν repetitions

Optimize over measurements ⇒ Quantum Fisher Info $\mathcal{I}(\rho(\theta))$

Cramér-Rao inequality $\Delta\bar{\theta} \geq 1/\sqrt{\nu\mathcal{I}(\rho(\theta))}$ with

$$\mathcal{I}(\rho(\theta)) = \lim_{d\theta\to 0} 8 \frac{1 - \mathrm{Tr}\sqrt{\sqrt{\rho(\theta)}\,\rho(\theta+d\theta)\sqrt{\rho(\theta)}}}{d\theta^2}$$

[1] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015) [arXiv:1502.07654].

Perturbative Approach to Parameter Estimation

Perturbative approach

Unitary transformation: $U(\theta) = U^{(0)} + \theta\, U^{(1)} + \theta^2\, U^{(2)} + \mathcal{O}(\theta^3)$

Total state

Transformed state: $|\tilde{\psi}\rangle = |\tilde{\psi}^{(0)}\rangle + \theta\,|\tilde{\psi}^{(1)}\rangle + \theta^2\,|\tilde{\psi}^{(2)}\rangle + \mathcal{O}(\theta^3)$

QFI: $\mathcal{I}(|\tilde{\psi}\rangle) = 4\Big(\langle\tilde{\psi}^{(1)}|\,\tilde{\psi}^{(1)}\rangle - |\langle\tilde{\psi}^{(0)}|\,\tilde{\psi}^{(1)}\rangle|^2\Big) + \mathcal{O}(\theta)$

Reduced state

after tracing: $\tilde{\rho}_k(\theta) = \tilde{\rho}_k^{(0)} + \theta\,\tilde{\rho}_k^{(1)} + \theta^2\,\tilde{\rho}_k^{(2)} + \mathcal{O}(\theta^3)$

Reduced state QFI:

$$\mathcal{I}\big(\tilde{\rho}_k(\theta)\big) = -4\,{}_k\langle\tilde{\psi}^{(0)}|\,\tilde{\rho}_k^{(2)}\,|\tilde{\psi}^{(0)}\rangle_k + \mathcal{O}(\theta)$$

Perturbative Approach to Parameter Estimation

Perturbative approach

Unitary transformation: $U(\theta) = U^{(0)} + \theta\, U^{(1)} + \theta^2\, U^{(2)} + \mathcal{O}(\theta^3)$

Total state

Transformed state: $|\tilde{\psi}\rangle = |\tilde{\psi}^{(0)}\rangle + \theta\, |\tilde{\psi}^{(1)}\rangle + \theta^2\, |\tilde{\psi}^{(2)}\rangle + \mathcal{O}(\theta^3)$

QFI: $\mathcal{I}(|\tilde{\psi}\rangle) = 4\Big(\langle\tilde{\psi}^{(1)}|\tilde{\psi}^{(1)}\rangle - |\langle\tilde{\psi}^{(0)}|\tilde{\psi}^{(1)}\rangle|^2\Big) + \mathcal{O}(\theta)$

Reduced state

after tracing: $\tilde{\rho}_k(\theta) = \tilde{\rho}_k^{(0)} + \theta\, \tilde{\rho}_k^{(1)} + \theta^2\, \tilde{\rho}_k^{(2)} + \mathcal{O}(\theta^3)$

Reduced state QFI:

$$\mathcal{I}\big(\tilde{\rho}_k(\theta)\big) = -4\,{}_k\langle\tilde{\psi}^{(0)}|\,\tilde{\rho}_k^{(2)}\,|\tilde{\psi}^{(0)}\rangle_k + \mathcal{O}(\theta)$$

Perturbative Approach to Parameter Estimation

Perturbative approach

Unitary transformation: $U(\theta) = U^{(0)} + \theta\, U^{(1)} + \theta^2\, U^{(2)} + \mathcal{O}(\theta^3)$

Total state

Transformed state: $|\tilde{\psi}\rangle = |\tilde{\psi}^{(0)}\rangle + \theta\, |\tilde{\psi}^{(1)}\rangle + \theta^2\, |\tilde{\psi}^{(2)}\rangle + \mathcal{O}(\theta^3)$

QFI: $\mathcal{I}(|\tilde{\psi}\rangle) = 4\Big(\langle\tilde{\psi}^{(1)}|\,\tilde{\psi}^{(1)}\rangle - |\langle\tilde{\psi}^{(0)}|\,\tilde{\psi}^{(1)}\rangle|^2\Big) + \mathcal{O}(\theta)$

Reduced state

after tracing: $\tilde{\rho}_k(\theta) = \tilde{\rho}_k^{(0)} + \theta\, \tilde{\rho}_k^{(1)} + \theta^2\, \tilde{\rho}_k^{(2)} + \mathcal{O}(\theta^3)$

Reduced state QFI:

$$\mathcal{I}\big(\tilde{\rho}_k(\theta)\big) = -4\, {}_k\langle\tilde{\psi}^{(0)}|\, \tilde{\rho}_k^{(2)}\, |\tilde{\psi}^{(0)}\rangle_k + \mathcal{O}(\theta)$$

Perturbative Approach to Parameter Estimation

Perturbative approach

Unitary transformation: $U(\theta) = U^{(0)} + \theta\, U^{(1)} + \theta^2\, U^{(2)} + \mathcal{O}(\theta^3)$

Total state

Transformed state: $|\tilde{\psi}\rangle = |\tilde{\psi}^{(0)}\rangle + \theta\, |\tilde{\psi}^{(1)}\rangle + \theta^2\, |\tilde{\psi}^{(2)}\rangle + \mathcal{O}(\theta^3)$

QFI: $\mathcal{I}(|\tilde{\psi}\rangle) = 4\Big(\langle\tilde{\psi}^{(1)}|\,\tilde{\psi}^{(1)}\rangle - |\langle\tilde{\psi}^{(0)}|\,\tilde{\psi}^{(1)}\rangle|^2\Big) + \mathcal{O}(\theta)$

Reduced state

after tracing: $\tilde{\rho}_k(\theta) = \tilde{\rho}_k^{(0)} + \theta\, \tilde{\rho}_k^{(1)} + \theta^2\, \tilde{\rho}_k^{(2)} + \mathcal{O}(\theta^3)$

Reduced state QFI:

$$\mathcal{I}\big(\tilde{\rho}_k(\theta)\big) = -4\, {}_k\langle\tilde{\psi}^{(0)}|\, \tilde{\rho}_k^{(2)}\, |\tilde{\psi}^{(0)}\rangle_k + \mathcal{O}(\theta)$$

Perturbative Approach to Parameter Estimation

Perturbative approach

Unitary transformation: $U(\theta) = U^{(0)} + \theta\, U^{(1)} + \theta^2\, U^{(2)} + \mathcal{O}(\theta^3)$

Total state

Transformed state: $|\tilde{\psi}\rangle = |\tilde{\psi}^{(0)}\rangle + \theta\, |\tilde{\psi}^{(1)}\rangle + \theta^2\, |\tilde{\psi}^{(2)}\rangle + \mathcal{O}(\theta^3)$

QFI: $\mathcal{I}(|\tilde{\psi}\rangle) = 4\Big(\langle\tilde{\psi}^{(1)}|\tilde{\psi}^{(1)}\rangle - |\langle\tilde{\psi}^{(0)}|\tilde{\psi}^{(1)}\rangle|^2\Big) + \mathcal{O}(\theta)$

Reduced state

after tracing: $\tilde{\rho}_k(\theta) = \tilde{\rho}_k^{(0)} + \theta\,\tilde{\rho}_k^{(1)} + \theta^2\,\tilde{\rho}_k^{(2)} + \mathcal{O}(\theta^3)$

Reduced state QFI:

$$\mathcal{I}\big(\tilde{\rho}_k(\theta)\big) = -4\,{}_k\langle\tilde{\psi}^{(0)}|\,\tilde{\rho}_k^{(2)}\,|\tilde{\psi}^{(0)}\rangle_k + \mathcal{O}(\theta)$$

Perturbative Approach to Parameter Estimation

Perturbative approach

Unitary transformation: $U(\theta) = U^{(0)} + \theta\, U^{(1)} + \theta^2\, U^{(2)} + \mathcal{O}(\theta^3)$

Total state

Transformed state: $|\tilde{\psi}\rangle = |\tilde{\psi}^{(0)}\rangle + \theta\, |\tilde{\psi}^{(1)}\rangle + \theta^2\, |\tilde{\psi}^{(2)}\rangle + \mathcal{O}(\theta^3)$

QFI: $\mathcal{I}(|\tilde{\psi}\rangle) = 4\Big(\langle\tilde{\psi}^{(1)}|\,\tilde{\psi}^{(1)}\rangle - |\langle\tilde{\psi}^{(0)}|\,\tilde{\psi}^{(1)}\rangle|^2\Big) + \mathcal{O}(\theta)$

Reduced state

after tracing: $\tilde{\rho}_k(\theta) = \tilde{\rho}_k^{(0)} + \theta\,\tilde{\rho}_k^{(1)} + \theta^2\,\tilde{\rho}_k^{(2)} + \mathcal{O}(\theta^3)$

Reduced state QFI:

$$\mathcal{I}\big(\tilde{\rho}_k(\theta)\big) = \mathcal{I}(|\tilde{\psi}\rangle) - 4\sum_{i\neq 0} |{}_k\langle\tilde{\psi}^{(0)}|_{\neg k}\langle i\,|\,U^{(1)}|\,\psi\rangle_k\,|0\rangle_{\neg k}|^2 + \mathcal{O}(\theta)$$

Specialize to Gaussian Transformations

Perturbative Bogoliubov transformation

Coefficients: $\alpha_{mn} = \alpha_{mn}^{(0)} + \alpha_{mn}^{(1)}\,\theta + \alpha_{mn}^{(2)}\,\theta^2 + O(\theta^3)$

$$\beta_{mn} = \beta_{mn}^{(1)}\,\theta + \beta_{mn}^{(2)}\,\theta^2 + O(\theta^3)$$

Phases (free time evolution): $\alpha_{mn}^{(0)} = \delta_{mn} G_n = \delta_{mn} \exp(i\phi_n)$

Vacuum: $|\,0\,\rangle \mapsto |\,0\,\rangle - \theta \frac{1}{2} \sum_{p,q} G_q^* \beta_{pq}^{(1)*} a_p^\dagger a_q^\dagger \,|\,0\,\rangle + \mathcal{O}(\theta^2)$

Tracing loss

QFI: $\mathcal{I}(\tilde{\rho}_k(\theta)) = \mathcal{I}(|\,\tilde{\psi}\,\rangle) - \Delta_{\mathrm{tr}}(|\,\psi\,\rangle_k) + \mathcal{O}(\theta)$

Tracing loss: $\Delta_{\mathrm{tr}}(|\,\psi\,\rangle_k) \geq \Delta_{\mathrm{tr}}(|\,0\,\rangle_k) = 2 \sum_{p,q \notin k} |\beta_{pq}^{(1)}|^2$

Specialize to Gaussian Transformations

Perturbative Bogoliubov transformation

Coefficients: $\alpha_{mn} = \alpha_{mn}^{(0)} + \alpha_{mn}^{(1)}\,\theta + \alpha_{mn}^{(2)}\,\theta^2 + O(\theta^3)$

$$\beta_{mn} = \beta_{mn}^{(1)}\,\theta + \beta_{mn}^{(2)}\,\theta^2 + O(\theta^3)$$

Phases (free time evolution): $\alpha_{mn}^{(0)} = \delta_{mn} G_n = \delta_{mn} \exp(i\phi_n)$

Vacuum: $|\,0\,\rangle \mapsto |\,0\,\rangle - \theta\frac{1}{2}\sum_{p,q} G_q^* \beta_{pq}^{(1)*} a_p^\dagger a_q^\dagger\,|\,0\,\rangle + \mathcal{O}(\theta^2)$

Tracing loss

QFI: $\mathcal{I}\big(\tilde{\rho}_k(\theta)\big) = \mathcal{I}(|\,\tilde{\psi}\,\rangle) - \Delta_{\text{tr}}(|\,\psi\,\rangle_k) + \mathcal{O}(\theta)$

Tracing loss: $\Delta_{\text{tr}}(|\,\psi\,\rangle_k) \geq \Delta_{\text{tr}}(|\,0\,\rangle_k) = 2\sum_{p,q\notin k} |\beta_{pq}^{(1)}|^2$

Specialize to Gaussian Transformations

Perturbative Bogoliubov transformation

Coefficients: $\alpha_{mn} = \alpha^{(0)}_{mn} + \alpha^{(1)}_{mn}\,\theta + \alpha^{(2)}_{mn}\,\theta^2 + O(\theta^3)$

$$\beta_{mn} = \beta^{(1)}_{mn}\,\theta + \beta^{(2)}_{mn}\,\theta^2 + O(\theta^3)$$

Phases (free time evolution): $\alpha^{(0)}_{mn} = \delta_{mn} G_n = \delta_{mn} \exp(i\phi_n)$

Vacuum: $|0\rangle \mapsto |0\rangle - \theta \frac{1}{2} \sum_{p,q} G^*_q \beta^{(1)*}_{pq} a^\dagger_p a^\dagger_q |0\rangle + \mathcal{O}(\theta^2)$

Tracing loss

QFI: $\mathcal{I}\big(\tilde{\rho}_k(\theta)\big) = \mathcal{I}(|\tilde{\psi}\rangle) - \Delta_{\mathrm{tr}}(|\psi\rangle_k) + \mathcal{O}(\theta)$

Tracing loss: $\Delta_{\mathrm{tr}}(|\psi\rangle_k) \geq \Delta_{\mathrm{tr}}(|0\rangle_k) = 2 \sum_{p,q \notin k} |\beta^{(1)}_{pq}|^2$

Specialize to Gaussian Transformations

Perturbative Bogoliubov transformation

Coefficients: $\alpha_{mn} = \alpha^{(0)}_{mn} + \alpha^{(1)}_{mn}\,\theta + \alpha^{(2)}_{mn}\,\theta^2 + O(\theta^3)$

$$\beta_{mn} = \beta^{(1)}_{mn}\,\theta + \beta^{(2)}_{mn}\,\theta^2 + O(\theta^3)$$

Phases (free time evolution): $\alpha^{(0)}_{mn} = \delta_{mn} G_n = \delta_{mn} \exp(i\phi_n)$

Vacuum: $|\,0\,\rangle \mapsto |\,0\,\rangle - \theta\frac{1}{2}\sum_{p,q} G^*_q \beta^{(1)*}_{pq} a^\dagger_p a^\dagger_q\,|\,0\,\rangle + \mathcal{O}(\theta^2)$

Tracing loss

QFI: $\mathcal{I}\big(\tilde{\rho}_k(\theta)\big) = \mathcal{I}(|\,\tilde{\psi}\,\rangle) - \Delta_{\mathrm{tr}}(|\,\psi\,\rangle_k) + \mathcal{O}(\theta)$

Tracing loss: $\Delta_{\mathrm{tr}}(|\,\psi\,\rangle_k) \geq \Delta_{\mathrm{tr}}(|\,0\,\rangle_k) = 2\sum_{p,q\notin k} |\beta^{(1)}_{pq}|^2$

Specialize to Gaussian Transformations

Perturbative Bogoliubov transformation

Coefficients: $\alpha_{mn} = \alpha_{mn}^{(0)} + \alpha_{mn}^{(1)}\,\theta + \alpha_{mn}^{(2)}\,\theta^2 + O(\theta^3)$

$$\beta_{mn} = \beta_{mn}^{(1)}\,\theta + \beta_{mn}^{(2)}\,\theta^2 + O(\theta^3)$$

Phases (free time evolution): $\alpha_{mn}^{(0)} = \delta_{mn} G_n = \delta_{mn} \exp(i\phi_n)$

Vacuum: $|0\rangle \mapsto |0\rangle - \theta\frac{1}{2}\sum_{p,q} G_q^* \beta_{pq}^{(1)*} a_p^\dagger a_q^\dagger |0\rangle + \mathcal{O}(\theta^2)$

Tracing loss

QFI: $\mathcal{I}\big(\tilde{\rho}_k(\theta)\big) = \mathcal{I}(|\tilde{\psi}\rangle) - \Delta_{\mathrm{tr}}(|\psi\rangle_k) + \mathcal{O}(\theta)$

Tracing loss: $\Delta_{\mathrm{tr}}(|\psi\rangle_k) \geq \Delta_{\mathrm{tr}}(|0\rangle_k) = 2\sum_{p,q\notin k} |\beta_{pq}^{(1)}|^2$

Optimal Scaling

- Have QFI for any initial state
- Know how to pick states with minimal tracing loss

e.g., $\mathcal{I}(|n_k\rangle) = 2n(n+1)|\beta^{(1)}_{kk}|^2 + 4n\sum_{p\neq k}\big(|\alpha^{(1)}_{pk}|^2 + |\beta^{(1)}_{pk}|^2\big) + \mathcal{I}(|0\rangle) + \mathcal{O}(\theta)$

or for $\beta^{(1)}_{kk} = \beta^{(1)}_{k'k'} = 0$

$$\mathcal{I}(|n_k\rangle|m_{k'}\rangle) = 8mn\big(|\alpha^{(1)}_{kk'}|^2 + |\beta^{(1)}_{kk'}|^2\big) + 4n\sum_{p\neq k}\big(|\alpha^{(1)}_{pk}|^2 + |\beta^{(1)}_{pk}|^2\big) + 4m\sum_{p\neq k'}\big(|\alpha^{(1)}_{pk'}|^2 + |\beta^{(1)}_{pk'}|^2\big) + \mathcal{I}(|0\rangle) + \mathcal{O}(\theta)$$

Heisenberg scaling optimal?

Optimal Scaling

- Have QFI for any initial state
- Know how to pick states with minimal tracing loss

e.g., $$\mathcal{I}(|n_k\rangle) = 2n(n+1)|\beta^{(1)}_{kk}|^2 + 4n\sum_{p\neq k}\left(|\alpha^{(1)}_{pk}|^2 + |\beta^{(1)}_{pk}|^2\right) + \mathcal{I}(|0\rangle) + \mathcal{O}(\theta)$$

or for $\beta^{(1)}_{kk} = \beta^{(1)}_{k'k'} = 0$

$$\mathcal{I}(|n_k\rangle|m_{k'}\rangle) = 8mn\left(|\alpha^{(1)}_{kk'}|^2 + |\beta^{(1)}_{kk'}|^2\right) + 4n\sum_{p\neq k}\left(|\alpha^{(1)}_{pk}|^2 + |\beta^{(1)}_{pk}|^2\right) + 4m\sum_{p\neq k'}\left(|\alpha^{(1)}_{pk'}|^2 + |\beta^{(1)}_{pk'}|^2\right) + \mathcal{I}(|0\rangle) + \mathcal{O}(\theta)$$

Heisenberg scaling optimal?

Optimal Scaling

- Have QFI for any initial state
- Know how to pick states with minimal tracing loss

e.g., $$\mathcal{I}(|n_k\rangle) = 2n(n+1)|\beta^{(1)}_{kk}|^2 + 4n\sum_{p\neq k}\big(|\alpha^{(1)}_{pk}|^2 + |\beta^{(1)}_{pk}|^2\big) + \mathcal{I}(|0\rangle) + \mathcal{O}(\theta)$$

or for $\beta^{(1)}_{kk} = \beta^{(1)}_{k'k'} = 0$

$$\mathcal{I}(|n_k\rangle|m_{k'}\rangle) = 8mn\big(|\alpha^{(1)}_{kk'}|^2 + |\beta^{(1)}_{kk'}|^2\big) + 4n\sum_{p\neq k}\big(|\alpha^{(1)}_{pk}|^2 + |\beta^{(1)}_{pk}|^2\big) + 4m\sum_{p\neq k'}\big(|\alpha^{(1)}_{pk'}|^2 + |\beta^{(1)}_{pk'}|^2\big) + \mathcal{I}(|0\rangle) + \mathcal{O}(\theta)$$

Heisenberg scaling optimal?

Optimal Scaling

- Have QFI for any initial state
- Know how to pick states with minimal tracing loss

$$\text{e.g., } \mathcal{I}(|n_k\rangle) = 2n(n+1)|\beta^{(1)}_{kk}|^2 + 4n \sum_{p\neq k} \big(|\alpha^{(1)}_{pk}|^2 + |\beta^{(1)}_{pk}|^2\big) + \mathcal{I}(|0\rangle) + \mathcal{O}(\theta)$$

or for $\beta^{(1)}_{kk} = \beta^{(1)}_{k'k'} = 0$

$$\mathcal{I}(|n_k\rangle|m_{k'}\rangle) = 8mn\big(|\alpha^{(1)}_{kk'}|^2 + |\beta^{(1)}_{kk'}|^2\big) + 4n \sum_{p\neq k} \big(|\alpha^{(1)}_{pk}|^2 + |\beta^{(1)}_{pk}|^2\big) + 4m \sum_{p\neq k'} \big(|\alpha^{(1)}_{pk'}|^2 + |\beta^{(1)}_{pk'}|^2\big) + \mathcal{I}(|0\rangle) + \mathcal{O}(\theta)$$

Heisenberg scaling optimal?

Optimal Scaling

- Have QFI for any initial state
- Know how to pick states with minimal tracing loss

e.g., $$\mathcal{I}(|n_k\rangle) = 2n(n+1)|\beta^{(1)}_{kk}|^2 + 4n\sum_{p\neq k}\left(|\alpha^{(1)}_{pk}|^2 + |\beta^{(1)}_{pk}|^2\right) + \mathcal{I}(|0\rangle) + \mathcal{O}(\theta)$$

or for $\beta^{(1)}_{kk} = \beta^{(1)}_{k'k'} = 0$

$$\mathcal{I}(|n_k\rangle|m_{k'}\rangle) = 8mn\left(|\alpha^{(1)}_{kk'}|^2 + |\beta^{(1)}_{kk'}|^2\right) + 4n\sum_{p\neq k}\left(|\alpha^{(1)}_{pk}|^2 + |\beta^{(1)}_{pk}|^2\right) + 4m\sum_{p\neq k'}\left(|\alpha^{(1)}_{pk'}|^2 + |\beta^{(1)}_{pk'}|^2\right) + \mathcal{I}(|0\rangle) + \mathcal{O}(\theta)$$

Heisenberg scaling optimal?

Optimal Scaling

- Have QFI for any initial state
- Know how to pick states with minimal tracing loss

e.g., $$\mathcal{I}(|n_k\rangle) = 2n(n+1)|\beta^{(1)}_{kk}|^2 + 4n\sum_{p\neq k}\big(|\alpha^{(1)}_{pk}|^2 + |\beta^{(1)}_{pk}|^2\big) + \mathcal{I}(|0\rangle) + \mathcal{O}(\theta)$$

or for $\beta^{(1)}_{kk} = \beta^{(1)}_{k'k'} = 0$

$$\mathcal{I}(|n_k\rangle|m_{k'}\rangle) = 8mn\big(|\alpha^{(1)}_{kk'}|^2 + |\beta^{(1)}_{kk'}|^2\big) + 4n\sum_{p\neq k}\big(|\alpha^{(1)}_{pk}|^2 + |\beta^{(1)}_{pk}|^2\big) + 4m\sum_{p\neq k'}\big(|\alpha^{(1)}_{pk'}|^2 + |\beta^{(1)}_{pk'}|^2\big) + \mathcal{I}(|0\rangle) + \mathcal{O}(\theta)$$

Heisenberg scaling optimal?

Optimal Scaling

- Have QFI for any initial state
- Know how to pick states with minimal tracing loss

e.g., $$\mathcal{I}(|n_k\rangle) = 2n(n+1)|\beta^{(1)}_{kk}|^2 + 4n\sum_{p\neq k}\big(|\alpha^{(1)}_{pk}|^2 + |\beta^{(1)}_{pk}|^2\big) + \mathcal{I}(|0\rangle) + \mathcal{O}(\theta)$$

or for $\beta^{(1)}_{kk} = \beta^{(1)}_{k'k'} = 0$

$$\mathcal{I}(|n_k\rangle|m_{k'}\rangle) = 8mn\big(|\alpha^{(1)}_{kk'}|^2 + |\beta^{(1)}_{kk'}|^2\big) + 4n\sum_{p\neq k}\big(|\alpha^{(1)}_{pk}|^2 + |\beta^{(1)}_{pk}|^2\big) + 4m\sum_{p\neq k'}\big(|\alpha^{(1)}_{pk'}|^2 + |\beta^{(1)}_{pk'}|^2\big) + \mathcal{I}(|0\rangle) + \mathcal{O}(\theta)$$

Heisenberg scaling optimal?

Optimal Scaling

OPTIMAL

Bogoliubov transformations in phase space

Gaussian States - covariance matrix formalism

$$\sigma_{ij} = \langle \mathrm{X}_i \mathrm{X}_j + \mathrm{X}_j \mathrm{X}_i \rangle - 2 \langle \mathrm{X}_i \rangle \langle \mathrm{X}_j \rangle$$

quadratures: $\mathrm{X}_{(2n-1)} = \frac{1}{\sqrt{2}}(a_n + a_n^\dagger)$ and $\mathrm{X}_{(2n)} = \frac{-i}{\sqrt{2}}(a_n - a_n^\dagger)$

symplectic form $\Omega_{mn} = -i\,[\mathrm{X}_m, \mathrm{X}_n]$

Bogoliubov transformation $\Rightarrow$ symplectic transformation S

$$\tilde{\sigma} = S\,\sigma\,S^T$$

$$S = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} & \cdots \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} & \cdots \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{where} \quad \mathcal{M}_{mn} = \begin{pmatrix} \mathrm{Re}(\alpha_{mn} - \beta_{mn}) & \mathrm{Im}(\alpha_{mn} + \beta_{mn}) \\ -\mathrm{Im}(\alpha_{mn} - \beta_{mn}) & \mathrm{Re}(\alpha_{mn} + \beta_{mn}) \end{pmatrix}$$

Bogoliubov transformations in phase space

Gaussian States - covariance matrix formalism

$$\sigma_{ij} = \langle \mathrm{X}_i\mathrm{X}_j + \mathrm{X}_j\mathrm{X}_i \rangle - 2\,\langle \mathrm{X}_i \rangle \langle \mathrm{X}_j \rangle$$

quadratures: $\mathrm{X}_{(2n-1)} = \frac{1}{\sqrt{2}}(a_n + a_n^\dagger)$ and $\mathrm{X}_{(2n)} = \frac{-i}{\sqrt{2}}(a_n - a_n^\dagger)$

symplectic form $\Omega_{mn} = -i\,[\mathrm{X}_m, \mathrm{X}_n]$

Bogoliubov transformation $\Rightarrow$ symplectic transformation S

$$\tilde{\sigma} = S\,\sigma\,S^T$$

$$S = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} & \cdots \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} & \cdots \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ where } \mathcal{M}_{mn} = \begin{pmatrix} \mathrm{Re}(\alpha_{mn} - \beta_{mn}) & \mathrm{Im}(\alpha_{mn} + \beta_{mn}) \\ -\mathrm{Im}(\alpha_{mn} - \beta_{mn}) & \mathrm{Re}(\alpha_{mn} + \beta_{mn}) \end{pmatrix}$$

Bogoliubov transformations in phase space

Gaussian States - covariance matrix formalism

$$\sigma_{ij} = \langle \mathrm{X}_i \mathrm{X}_j + \mathrm{X}_j \mathrm{X}_i \rangle - 2 \langle \mathrm{X}_i \rangle \langle \mathrm{X}_j \rangle$$

quadratures: $\mathrm{X}_{(2n-1)} = \frac{1}{\sqrt{2}}(a_n + a_n^\dagger)$ and $\mathrm{X}_{(2n)} = \frac{-i}{\sqrt{2}}(a_n - a_n^\dagger)$

symplectic form $\Omega_{mn} = -i\,[\mathrm{X}_m, \mathrm{X}_n]$

Bogoliubov transformation $\Rightarrow$ symplectic transformation S

$$\tilde{\sigma} = S\,\sigma\,S^T$$

$$S = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} & \cdots \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} & \cdots \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{where} \quad \mathcal{M}_{mn} = \begin{pmatrix} \mathrm{Re}(\alpha_{mn} - \beta_{mn}) & \mathrm{Im}(\alpha_{mn} + \beta_{mn}) \\ -\mathrm{Im}(\alpha_{mn} - \beta_{mn}) & \mathrm{Re}(\alpha_{mn} + \beta_{mn}) \end{pmatrix}$$

Bogoliubov transformations in phase space

Gaussian States - covariance matrix formalism

$$\sigma_{ij} = \langle \mathrm{X}_i \mathrm{X}_j + \mathrm{X}_j \mathrm{X}_i \rangle - 2 \langle \mathrm{X}_i \rangle \langle \mathrm{X}_j \rangle$$

quadratures: $\mathrm{X}_{(2n-1)} = \frac{1}{\sqrt{2}}(a_n + a_n^\dagger)$ and $\mathrm{X}_{(2n)} = \frac{-i}{\sqrt{2}}(a_n - a_n^\dagger)$

symplectic form $\Omega_{mn} = -i\,[\mathrm{X}_m, \mathrm{X}_n]$

Bogoliubov transformation ⇒ symplectic transformation S

$$\tilde{\sigma} = S\,\sigma\,S^T$$

$$S = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} & \cdots \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} & \cdots \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ where } \mathcal{M}_{mn} = \begin{pmatrix} \mathrm{Re}(\alpha_{mn} - \beta_{mn}) & \mathrm{Im}(\alpha_{mn} + \beta_{mn}) \\ -\mathrm{Im}(\alpha_{mn} - \beta_{mn}) & \mathrm{Re}(\alpha_{mn} + \beta_{mn}) \end{pmatrix}$$

Tracing & Fidelity

Transforming initial state

$$\sigma = \begin{pmatrix} C_{11} & C_{12} & C_{13} & \cdots \\ C_{21} & C_{22} & C_{23} & \cdots \\ C_{31} & C_{32} & C_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \Rightarrow \tilde{\sigma} = \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} & \cdots \\ \tilde{C}_{21} & \tilde{C}_{22} & \tilde{C}_{23} & \cdots \\ \tilde{C}_{31} & \tilde{C}_{32} & \tilde{C}_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $\tilde{C}_{mn} = \sum_{i,j} M_{mi} C_{ij} M^T_{nj}$

[a] P. Marian and T. A. Marian, Phys. Rev. A 86, 022340 (2012) [arXiv:1111.7067].

Tracing & Fidelity

Transforming initial state

$$\sigma = \begin{pmatrix} C_{11} & C_{12} & C_{13} & \cdots \\ C_{21} & C_{22} & C_{23} & \cdots \\ C_{31} & C_{32} & C_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \Rightarrow \tilde{\sigma} = \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} & \cdots \\ \tilde{C}_{21} & \tilde{C}_{22} & \tilde{C}_{23} & \cdots \\ \tilde{C}_{31} & \tilde{C}_{32} & \tilde{C}_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $\tilde{C}_{mn} = \sum_{i,j} \mathcal{M}_{mi} C_{ij} \mathcal{M}^T_{nj}$

[a] P. Marian and T. A. Marian, Phys. Rev. A 86, 022340 (2012) [arXiv:1111.7067]

Tracing & Fidelity

Tracing out inaccessible modes, e.g., all but 2 & 3

$$\sigma = \begin{pmatrix} C_{11} & C_{12} & C_{13} & \cdots \\ C_{21} & C_{22} & C_{23} & \cdots \\ C_{31} & C_{32} & C_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \Rightarrow \tilde{\sigma} = \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} & \cdots \\ \tilde{C}_{21} & \tilde{C}_{22} & \tilde{C}_{23} & \cdots \\ \tilde{C}_{31} & \tilde{C}_{32} & \tilde{C}_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $\tilde{C}_{mn} = \sum_{i,j} \mathcal{M}_{mi} C_{ij} \mathcal{M}^T_{nj}$

Fidelity [3] (e.g., for 2 modes)

$$\mathcal{F}(\rho, \rho') = \exp\Big(-\tfrac{1}{2}\delta\langle X \rangle^T (\sigma + \sigma')^{-1} \delta\langle X \rangle\Big)$$
$$\times \Big[(\sqrt{\Gamma} + \sqrt{\Lambda}) + \sqrt{(\sqrt{\Gamma} + \sqrt{\Lambda})^2 - \Delta}\Big]^{-1}$$

[3] P. Marian and T. A. Marian, Phys. Rev. A **86**, 022340 (2012) [arXiv:1111.7067].

Tracing & Fidelity

Tracing out inaccessible modes, e.g., all but 2 & 3

$$\sigma = \begin{pmatrix} C_{11} & C_{12} & C_{13} & \cdots \\ C_{21} & C_{22} & C_{23} & \cdots \\ C_{31} & C_{32} & C_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \Rightarrow \tilde{\sigma} = \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} & \cdots \\ \tilde{C}_{21} & \tilde{C}_{22} & \tilde{C}_{23} & \cdots \\ \tilde{C}_{31} & \tilde{C}_{32} & \tilde{C}_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $\tilde{C}_{mn} = \sum_{i,j} \mathcal{M}_{mi} C_{ij} \mathcal{M}^T_{nj}$

Fidelity [3] (e.g., for 2 modes)

$$\mathcal{F}(\rho, \rho') = \exp\Big(-\tfrac{1}{2}\delta\langle\, X\,\rangle^T (\sigma + \sigma')^{-1} \delta\langle\, X\,\rangle\Big)$$

$$\times\Big[(\sqrt{\Gamma} + \sqrt{\Lambda}) + \sqrt{(\sqrt{\Gamma} + \sqrt{\Lambda})^2 - \Delta}\Big]^{-1}$$

where $\Delta = \det(\sigma + \sigma')$, $\Gamma = 2^{2n}\det\Big((\Omega\sigma)(\Omega\sigma') - \tfrac{1}{4}\mathbb{1}\Big)$

and $\Lambda = 2^{2n}\det(\sigma + \tfrac{i}{2}\Omega)\det(\sigma' + \tfrac{i}{2}\Omega)$

[3] P. Marian and T. A. Marian, Phys. Rev. A **86**, 022340 (2012) [arXiv:1111.7067].

Tracing & Fidelity

Tracing out inaccessible modes, e.g., all but 2 & 3

$$\sigma = \begin{pmatrix} C_{11} & C_{12} & C_{13} & \dots \\ C_{21} & C_{22} & C_{23} & \dots \\ C_{31} & C_{32} & C_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \Rightarrow \tilde{\sigma} = \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} & \dots \\ \tilde{C}_{21} & \tilde{C}_{22} & \tilde{C}_{23} & \dots \\ \tilde{C}_{31} & \tilde{C}_{32} & \tilde{C}_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $\tilde{C}_{mn} = \sum_{i,j} \mathcal{M}_{mi} C_{ij} \mathcal{M}^T_{nj}$

Fidelity [3] (e.g., for 2 modes)

$$\mathcal{F}(\rho,\rho') = \exp\Big(-\tfrac{1}{2}\delta\langle\,\mathrm{X}\,\rangle^T(\sigma+\sigma')^{-1}\delta\langle\,\mathrm{X}\,\rangle\Big)$$
$$\times\Big[(\sqrt{\Gamma}+\sqrt{\Lambda})+\sqrt{(\sqrt{\Gamma}+\sqrt{\Lambda})^2-\Delta}\Big]^{-1}$$

where $\Delta = \det(\sigma+\sigma')$, $\Gamma = 2^{2n}\det\Big((\Omega\sigma)(\Omega\sigma')-\tfrac{1}{4}\mathbb{1}\Big)$

and $\Lambda = 2^{2n}\det\big(\sigma+\tfrac{i}{2}\Omega\big)\det\big(\sigma'+\tfrac{i}{2}\Omega\big)$

[3] P. Marian and T. A. Marian, Phys. Rev. A **86**, 022340 (2012) [arXiv:1111.7067].

Tracing & Fidelity

Tracing out inaccessible modes, e.g., all but 2 & 3

$$\sigma = \begin{pmatrix} C_{11} & C_{12} & C_{13} & \cdots \\ C_{21} & C_{22} & C_{23} & \cdots \\ C_{31} & C_{32} & C_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \Rightarrow \tilde{\sigma} = \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} & \cdots \\ \tilde{C}_{21} & \tilde{C}_{22} & \tilde{C}_{23} & \cdots \\ \tilde{C}_{31} & \tilde{C}_{32} & \tilde{C}_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $\tilde{C}_{mn} = \sum_{i,j} \mathcal{M}_{mi} C_{ij} \mathcal{M}^T_{nj}$

Fidelity [3] (e.g., for 2 modes)

$$\mathcal{F}(\rho,\rho') = \exp\Big(-\tfrac{1}{2}\delta\langle\, \mathrm{X}\,\rangle^T (\sigma+\sigma')^{-1}\delta\langle\, \mathrm{X}\,\rangle\Big)$$
$$\times\Big[(\sqrt{\Gamma}+\sqrt{\Lambda}) + \sqrt{(\sqrt{\Gamma}+\sqrt{\Lambda})^2 - \Delta}\Big]^{-1}$$

where $\Delta = \det(\sigma+\sigma')$, $\Gamma = 2^{2n}\det\Big((\Omega\sigma)(\Omega\sigma') - \frac{1}{4}\mathbb{1}\Big)$
and $\Lambda = 2^{2n}\det\big(\sigma + \frac{i}{2}\Omega\big)\det\big(\sigma' + \frac{i}{2}\Omega\big)$

[3] P. Marian and T. A. Marian, Phys. Rev. A **86**, 022340 (2012) [arXiv:1111.7067].

Some results for Gaussian states

Optimal Gaussian states

- R. Gaiba and M. G. A. Paris, Phys. Lett. A **373**, 934 (2009) [arXiv:0802.1682].

One-mode and two-mode Gaussian state QFI

- M. Ahmadi, D. E. Bruschi, C. Sabín, G. Adesso, and I. Fuentes, Sci. Rep. **4**, 4996 (2014) [arXiv:1307.7082].
- M. Ahmadi, D. E. Bruschi, and I. Fuentes, Phys. Rev. D **89**, 065028 (2014) [arXiv:1312.5707].
- D. Šafránek, A. R. Lee, and I. Fuentes, New J. Phys. **17**, 073016 (2015) [arXiv:1502.07924].

Some results for Gaussian states

Optimal Gaussian states

- R. Gaiba and M. G. A. Paris, Phys. Lett. A **373**, 934 (2009) [arXiv:0802.1682].

One-mode and two-mode Gaussian state QFI

- M. Ahmadi, D. E. Bruschi, C. Sabín, G. Adesso, and I. Fuentes, Sci. Rep. **4**, 4996 (2014) [arXiv:1307.7082].
- M. Ahmadi, D. E. Bruschi, and I. Fuentes, Phys. Rev. D **89**, 065028 (2014) [arXiv:1312.5707].
- D. Šafránek, A. R. Lee, and I. Fuentes, New J. Phys. **17**, 073016 (2015) [arXiv:1502.07924].

Summary (for proofs please see [1])

- Parameter estimation framework for perturbative Bogoliubov transformations
- Works for any initial states (not just Gaussian, e.g., as in [3])
- Quantify & minimize tracing losses
- Show that Heisenberg scaling is optimal
- Determine family of ALL optimally scaling & minimal tracing loss states

[1] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015) [arXiv:1502.07654].

[3] M. Ahmadi, D. E. Bruschi, and I. Fuentes, Phys. Rev. D **89**, 065028 (2014) [arXiv:1312.5707].

Summary (for proofs please see [1])

- Parameter estimation framework for perturbative Bogoliubov transformations
- Works for any initial states (not just Gaussian, e.g., as in [3])
- Quantify & minimize tracing losses
- Show that Heisenberg scaling is optimal
- Determine family of ALL optimally scaling & minimal tracing loss states

[1] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015) [arXiv:1502.07654].

[3] M. Ahmadi, D. E. Bruschi, and I. Fuentes, Phys. Rev. D **89**, 065028 (2014) [arXiv:1312.5707].

Summary (for proofs please see [1])

- Parameter estimation framework for perturbative Bogoliubov transformations
- Works for any initial states (not just Gaussian, e.g., as in [3])
- Quantify & minimize tracing losses
- Show that Heisenberg scaling is optimal
- Determine family of ALL optimally scaling & minimal tracing loss states

[1] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015) [arXiv:1502.07654].

[3] M. Ahmadi, D. E. Bruschi, and I. Fuentes, Phys. Rev. D **89**, 065028 (2014) [arXiv:1312.5707].

Summary (for proofs please see [1])

- Parameter estimation framework for perturbative Bogoliubov transformations
- Works for any initial states (not just Gaussian, e.g., as in [3])
- Quantify & minimize tracing losses
- Show that Heisenberg scaling is optimal
- Determine family of ALL optimally scaling & minimal tracing loss states

[1] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015) [arXiv:1502.07654].

[3] M. Ahmadi, D. E. Bruschi, and I. Fuentes, Phys. Rev. D **89**, 065028 (2014) [arXiv:1312.5707].

Summary (for proofs please see [1])

- Parameter estimation framework for perturbative Bogoliubov transformations
- Works for any initial states (not just Gaussian, e.g., as in [3])
- Quantify & minimize tracing losses
- Show that Heisenberg scaling is optimal
- Determine family of ALL optimally scaling & minimal tracing loss states

[1] N. Friis, M. Skotiniotis, I. Fuentes, and W. Dür, Phys. Rev. A **92**, 022106 (2015) [arXiv:1502.07654].

[3] M. Ahmadi, D. E. Bruschi, and I. Fuentes, Phys. Rev. D **89**, 065028 (2014) [arXiv:1312.5707].

Thank you for your Attention