Conditional mutual information in infinite-dimensional quantum systems and its use

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Let $H(\rho) = \mathrm{Tr}\eta(\rho) - \eta(\mathrm{Tr}\rho)$ be the extension of the von Neumann entropy to the cone $\mathfrak{T}_+(\mathcal{H})$ s.t. $H(\lambda\rho) = \lambda H(\rho)$ ($\eta(x) = -x\log x$)

$$I(A\!:\!C|B)_\omega \doteq H(\omega_{AB}) + H(\omega_{BC}) - H(\omega_{ABC}) - H(\omega_B). \qquad (1)$$

Basic properties:

1) $I(A:C|B)_\omega \geq 0$ for any state ω_{ABC} and $I(A:C|B)_\omega = 0$ if and only if there is a channel $\Phi : B \to BC$ such that $\omega_{ABC} = \mathrm{Id}_A \otimes \Phi(\omega_{AB})$;

2) $I(A:C|B)_\omega \geq I(A':C'|B)_{\Phi_A\otimes \mathrm{Id}_B\otimes \Phi_C(\omega)}$ for arbitrary quantum operations $\Phi_A : A \to A'$ and $\Phi_C : C \to C'$;

3) monotonicity under loc. conditioning: $I(AB:C)_\omega \geq I(A:C|B)_\omega$

4) additivity: $I(AA':CC'|BB')_{\omega\otimes\omega'} = I(A:C|B)_\omega + I(A':C'|B')_{\omega'}$;

5) duality: $I(A:C|B)_\omega = I(A:C|D)_\omega$ for any pure state ω_{ABCD}.

Operational meaning: communication cost of the quantum state redistribution protocol [I.Devetak, J.Yard, Phys. Rev. Lett. 100, 230501 (2008)]

Question: How to define $I(A:C|B)_\omega$ for states with infinite marginal entropies?

Motivating example: the quantum mutual information

$$I(A\!:\!B)_\omega \doteq H(\omega_A) + H(\omega_B) - H(\omega_{AB})$$

is well-defined for any state ω_{AB} by the formula

$$I(A\!:\!B)_\omega \doteq H(\omega_{AB}\,\|\,\omega_A \otimes \omega_B)$$

Properties of the relative entropy show that $\omega_{AB} \mapsto I(A:B)_\omega$ is a lower semicontinuous function on $\mathfrak{S}(\mathcal{H}_{AB})$ taking values in $[0,+\infty]$ and possessing all basic properties of mutual information.

Partial answers:

$$I(A\!:\!C|B)_\omega = I(A\!:\!BC)_\omega - I(A\!:\!B)_\omega, \qquad I(A\!:\!B)_\omega < +\infty \tag{2}$$

$$I(A\!:\!C|B)_\omega = I(AB\!:\!C)_\omega - I(B\!:\!C)_\omega, \qquad I(B\!:\!C)_\omega < +\infty \tag{3}$$

$$I(A\!:\!C|B)_\omega = I(A\!:\!C)_\omega - I(A\!:\!B)_\omega - I(C\!:\!B)_\omega + I(AC\!:\!B)_\omega, \tag{4}$$

$$H(\omega_B) < +\infty$$

$$\begin{aligned} I(A\!:\!C|B)_\omega = {} & I(A\!:\!C)_\omega + I(AB\!:\!D)_{\tilde{\omega}} + I(BC\!:\!D)_{\tilde{\omega}} \\ & + I(AC\!:\!D)_{\tilde{\omega}} - 4H(\omega_{ABC}), \quad H(\omega_{ABC}) < +\infty, \end{aligned} \tag{5}$$

where $\tilde{\omega}_{ABCD}$ is any purification of the state ω_{ABC}.

Question: Do formulas (2)-(5) agree with each other?

Let $\mathfrak{S}_{\mathrm{X}}$ be a subset of $\mathfrak{S}(\mathcal{H}_{ABC})$ where formula (X) is well defined.

Theorem 1. There exists a unique lower semicontinuous function $I_{\mathrm{e}}(A\!:\!C|B)_{\omega}$ on the set $\mathfrak{S}(\mathcal{H}_{ABC})$ such that:

- $I_{\mathrm{e}}(A\!:\!C|B)_{\omega}$ coincides with $I(A\!:\!C|B)_{\omega}$ given by (1),(2), (3), (4), (5) respectively on the sets $\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \mathfrak{S}_4, \mathfrak{S}_5$;

- $I_{\mathrm{e}}(A : C|B)_{\omega}$ possesses the above-stated properties 1-5 of conditional mutual information.

This function can be defined by one of the equivalent expressions

$$I_{\mathrm{e}}(A\!:\!C|B)_\omega = \sup_{P_A}\left[I(A\!:\!BC)_{Q\omega Q} - I(A\!:\!B)_{Q\omega Q}\right],\ \ Q = P_A\otimes I_B\otimes I_C,$$

$$I_{\mathrm{e}}(A\!:\!C|B)_\omega = \sup_{P_C}\left[I(AB\!:\!C)_{Q\omega Q} - I(B\!:\!C)_{Q\omega Q}\right],\ \ Q = I_A\otimes I_B\otimes P_C,$$

where the suprema are over all finite rank projectors $P_X \in \mathfrak{B}(\mathcal{H}_X)$.

For an arbitrary state $\omega \in \mathfrak{S}(\mathcal{H}_{ABCD})$ the following property is valid:

$$I_{\mathrm{e}}(A\!:\!C|B)_\omega = \lim_{k\to\infty}\lim_{l\to\infty} I(A\!:\!C|B)_{\omega^{kl}},$$

where

$$\omega^{kl} = \lambda_{kl}^{-1} Q_{kl}\,\omega\, Q_{kl}, \quad Q_{kl} = P_A^k \otimes P_B^l \otimes P_C^k \otimes P_D^l,\ \lambda_{kl} = \mathrm{Tr} Q_{kl}\omega.$$

[Theorem 2,Corollary 9 in arxiv:1506.06377]

Short Markov chains and recovery maps

Th.1 $\Rightarrow$ $\{\,\omega_{ABC}\,|\,I(A\!:\!C|B)_\omega = 0\,\}$ is a closed subset of $\mathfrak{S}(\mathcal{H}_{ABC})$.

If $I(A\!:\!B)_\omega$ is finite then the existence of a channel $\Phi : B \to BC$ such that $\omega_{ABC} = \mathrm{Id}_A \otimes \Phi(\omega_{AB})$ follows from Petz's theorem [P.Hayden, R.Jozsa, D.Petz, A.Winter, CMP 246:2, 359-374].

If $I(A\!:\!B)_\omega = +\infty$ then the existence of a recovery channel is proved by using the compactness criterion:

A sequence $\{\Phi_n\}$ of quantum operations $A \to B$ is relatively compact in the strong convergence topology if there is a full rank state ρ_A such that the sequence $\{\Phi_n(\rho_A)\}$ is relatively compact.

[A.S.Holevo,M.E.Shirokov, arXiv:0711.2245]

Open question: geometric structure of short Markov chains.

On existence of the Fawzi-Renner recovery channel for all states

For any state ω_{ABC} with finite marginal entropies there exists a recovery channel $\Phi : B \to BC$ such that

$$2^{-\frac{1}{2}I(A:C|B)_{\omega}} \leq F(\omega_{ABC}, \mathrm{Id}_A \otimes \Phi(\omega_{AB}))$$

where $F(\rho,\sigma) \doteq \|\sqrt{\rho}\sqrt{\sigma}\|_1$ is the quantum fidelity between states ρ and σ. [O.Fawzi, R.Renner arXiv:1410.0664, D.Sutter, O.Fawzi, R.Renner arXiv:1504.07251].

The above compactness criterion and the lower semicontinuity of $I(A:C|B)_{\omega}$ make possible to show the existence of Fawzi-Renner channel Φ such that $[\Phi(\omega_B)]_B = \omega_B$ and $[\Phi(\omega_B)]_C = \omega_C$ for arbitrary state ω_{ABC} starting from the corresponding finite-dimensional result in arXiv:1410.0664. [Prop.4 in arxiv:1506.06377]

Corollaries of the lower semicontinuity of $I(A\!:\!C|B)$

Corollary 1. Local continuity of one of the marginal entropies

$$H(\omega_A),\ H(\omega_C),\ H(\omega_{AB}),\ H(\omega_{AC})$$

implies local continuity of $I(A\!:\!C|B)_\omega$, i.e.

$$\lim_{n\to\infty} H(\omega^n_X) = H(\omega^0_X) < +\infty \ \Rightarrow\ \lim_{n\to\infty} I(A\!:\!C|B)_{\omega^n} = I(A\!:\!C|B)_{\omega^0}$$

$X = A, C, AB, BC$, for any sequence $\omega^n_{ABC} \to \omega^0_{ABC}$.

Corollary 2. Let $\omega^n_{AB} \to \omega^0_{AB}$ and there exists $\omega^n_{ABE} \to \omega^0_{ABE}$ s.t.

$$\lim_{n\to\infty} H(\omega^n_{AE}) = H(\omega^0_{AE}) < +\infty \ \text{ then } \ \lim_{n\to\infty} I(A\!:\!B)_{\omega^n} \to I(A\!:\!B)_{\omega^0}$$

Corollary 3. For any q. operations $\Phi : A \to A'$ and $\Psi : B \to B'$ the nonnegative function $\omega_{AB} \mapsto \left[I(A\!:\!B)_\omega - I(A'\!:\!B')_{\Phi\otimes\Psi(\omega)} \right]$ is lower semicontinuous.

Local continuity of the function $\omega_{AB} \mapsto I(A:B)_\omega$ implies local continuity of the function $\omega_{AB} \mapsto I(A'\!:\!B')_{\Phi\otimes\Psi(\omega)}$.

Example: Let $\{\omega^n_{AB}\}$ be a sequence of Gaussian states with bounded energy of A converging to a state ω^0_{AB} then

$$\lim_{n\to\infty} I(A'\!:\!B')_{\Phi\otimes\Psi(\omega^n)} = I(A'\!:\!B')_{\Phi\otimes\Psi(\omega^0)}$$

for arbitrary quantum channels $\Phi : A \to A'$ and $\Psi : B \to B'$.

The above results are valid for $I(A\!:\!B|C)$ (instead of $I(A\!:\!B)$).

A sequence $\{\{p_i^n, \rho_i^n\}\}_n$ converges to an ensemble $\{p_i^0, \rho_i^0\}$ if

$$\lim_{n\to\infty} p_i^n = p_i^0 \quad \forall i \quad \text{and} \quad \lim_{n\to\infty} \rho_i^n = \rho_i^0 \quad \forall i : p_i^0 \neq 0.$$

The Holevo quantity

$$\chi(\{p_i, \rho_i\}) = I(A\!:\!B)_{\widehat{\omega}}, \quad \text{where} \quad \widehat{\omega}_{AB} = \textstyle\sum_i p_i \rho_i \otimes |i\rangle\langle i|.$$

Corollary 4. For any channel $\Phi : A \to A'$ the nonnegative function $\{p_i, \rho_i\} \mapsto [\,\chi(\{p_i, \rho_i\}) - \chi(\{p_i, \Phi(\rho_i)\})]$ is lower semicontinuous on the set of all countable ensembles of states in $\mathfrak{S}(\mathcal{H}_A)$.

Local continuity of $\chi(\{p_i, \rho_i\})$ implies local continuity of $\chi(\{p_i, \Phi(\rho_i)\})$:

$$\chi(\{p_i^n, \rho_i^n\}) \to \chi(\{p_i^0, \rho_i^0\}) < +\infty \;\; \Rightarrow \;\; \chi(\{p_i^n, \Phi(\rho_i^n)\}) \to \chi(\{p_i^0, \Phi(\rho_i^0)\})$$

for any sequence $\{\{p_i^n, \rho_i^n\}\}_n$ converging to an ensemble $\{p_i^0, \rho_i^0\}$.

Different continuity bounds for $I(A\!:\!C|B)$ and their use.

Lemma 1. Let V_A be an operator in $\mathfrak{B}(\mathcal{H}_A)$ such that $\|V_A\| \leq 1$ and ω_{ABC} be a state with finite $H(\omega_A)$. Then

$$0 \leq I(A\!:\!C|B)_\omega - I(A\!:\!C|B)_{\tilde{\omega}} \leq 2\left[H(\omega_A) - H(V_A\omega_A V_A^*)\right],$$

and hence

$$-2\delta H(V_A\omega_A V_A^*) \leq I(A\!:\!C|B)_\omega - I(A\!:\!C|B)_{\frac{\tilde{\omega}}{\mathrm{Tr}\tilde{\omega}}} \leq 2\left[H(\omega_A) - H(V_A\omega_A V_A^*)\right],$$

where $\tilde{\omega}_{ABC} = V_A \otimes I_{BC}\, \omega_{ABC} V_A^* \otimes I_{BC}$ and $\delta = \frac{1-\mathrm{Tr}\tilde{\omega}}{\mathrm{Tr}\tilde{\omega}}$.

[Lemma 9 in arXiv:1507.08964]

Winter's type continuity bound for $I(A\!:\!C|B)$.

Let H_A be the Hamiltonian of system A such that Tr$e^{-\beta H_A} < +\infty$ for all $\beta > 0$ and $\gamma(E)$ is the Gibbs state for energy E.

Winter's technique [arXiv:1507.07775] + Lemma 1 =

Proposition 1. Let ρ_{ABC} and σ_{ABC} be states s.t. Tr$H_A\rho_A \leq E$, Tr$H_A\sigma_A \leq E$, $\frac{1}{2}\|\rho-\sigma\|_1 \leq \varepsilon < \varepsilon' \leq 1$ and $\delta = \frac{\varepsilon'-\varepsilon}{1+\varepsilon'}$. Then

$$|I(A\!:\!C|B)_\rho - I(A\!:\!C|B)_\sigma| \leq (2\varepsilon'+4\delta)H(\gamma(E/\delta))+2g(\varepsilon')+4h_2(\delta),$$

where $g(x) = (1+x)h_2\left(\frac{x}{1+x}\right) = (x+1)\log(x+1) - x\log x$.

This continuity bound is asymptotically tight (for large E) even for trivial B, i.e. in the case $I(A:C|B) = I(A:C)$. Since $\lim_{\delta\to 0}\delta\gamma(E/\delta) = 0$, it implies uniform continuity of $I(A\!:\!C|B)$ on the set of states with bounded energy of A.

Continuity bounds for E_{sq} and for E_F under energy constraints.

Corollary 5. Let ω^1_{AB} and ω^2_{AB} be states s.t. $\mathrm{Tr} H_A \omega^k_A \leq E$, $k = 1, 2$, and $\|\omega^2_{AB} - \omega^1_{AB}\|_1 = \varepsilon$ and Let $\varepsilon' \in (\sqrt{\varepsilon}, 1]$ and $\delta = \frac{\varepsilon' - \sqrt{\varepsilon}}{1+\varepsilon'}$. Then

$$\left| E_{sq}(\omega^2_{AB}) - E_{sq}(\omega^1_{AB}) \right| \leq (\varepsilon' + 2\delta) H(\gamma(E/\delta)) + g(\varepsilon') + 2h_2(\delta)$$

The same continuity bound is valid for the entanglement of formation E_F.

Since $\lim\limits_{\delta \to 0} \delta\gamma(E/\delta) = 0$, Corollary 5 implies asymptotic continuity of E_{sq} and E_F under the energy constrain on one subsystem (see details in [arXiv:1507.08964]).

Special continuity bound for $I(A\!:\!B|C)$.

By using the Alicki-Fannes-Winter technic one can obtain the following lemma (in which A, B and C are arbitrary systems).

Lemma 2. Let ρ_{ABC} and σ_{ABC} be states having extensions $\hat{\rho}_{ABCE}$ and $\hat{\sigma}_{ABCE}$ such that $\hat{\rho}_{AE}$ and $\hat{\sigma}_{AE}$ are finite rank states. Then $I(A\!:\!B|C)_\rho$ and $I(A\!:\!B|C)_\sigma$ are finite and

$$|I(A\!:\!B|C)_\rho - I(A\!:\!B|C)_\sigma| \leq 2\varepsilon \log d + 2g(\varepsilon), \tag{6}$$

where $d \doteq \dim(\operatorname{supp}\hat{\rho}_{AE} \vee \operatorname{supp}\hat{\sigma}_{AE})$ and $\varepsilon = \frac{1}{2}\|\hat{\rho} - \hat{\sigma}\|_1$.

If $\hat{\rho}$ and $\hat{\sigma}$ are qc-states with respect to the decomposition $(AE)(BC)$ then the factor 2 in (6) can be removed.

Lemma 2 makes possible to obtain estimates for variation of information characteristic of quantum channels depending on their input dimension.

The Bures distance: $\beta(\Phi,\Psi)=\inf\|V_{\Phi}-V_{\Psi}\|$, where the infimum is over all common Stinespring representations:

$$\Phi(\rho)=\mathrm{Tr}_{E}V_{\Phi}\rho V_{\Phi}^{*}\quad\text{and}\quad\Psi(\rho)=\mathrm{Tr}_{E}V_{\Psi}\rho V_{\Psi}^{*}.$$

The Bures distance is equivalent to the diamond norm distance:

$$\tfrac{1}{2}\|\Phi-\Psi\|_{\diamond}\leq\beta(\Phi,\Psi)\leq\sqrt{\|\Phi-\Psi\|_{\diamond}},$$

[D.Kretschmann, D.Schlingemann, R.F.Werner, arXiv:0710.2495]

Proposition 2. Let $\Phi : A \to B$ and $\Psi : A \to B$ be arbitrary quantum channels and C be any system. Let ρ_{AC} and σ_{AC} be states with finite rank marginals ρ_A and σ_A. Then

$$|I(B\!:\!C)_{\Phi\otimes\mathrm{Id}_C(\rho)} - I(B\!:\!C)_{\Psi\otimes\mathrm{Id}_C(\sigma)}| \leq 2\varepsilon\log(2d_A) + 2g(\varepsilon), \tag{7}$$

where $d_A = \dim(\mathrm{supp}\,\rho_A \vee \mathrm{supp}\,\sigma_A)$, $\varepsilon = \frac{1}{2}\|\rho-\sigma\|_1 + \beta(\Phi,\Psi)$.

If $\Phi = \Psi$ then the factor 2 in (7) can be removed.

If ρ_{AC} and σ_{AC} are qc-states then the first factor 2 in (7) can be removed.

Continuity bound (7) is tight in the both cases $\Phi = \Psi$ and $\rho = \sigma$. The Bures distance $\beta(\Phi,\Psi)$ in (7) can be replaced by $\|\Phi-\Psi\|_\diamond^{1/2}$.

Corollary 6. Let $\Phi : A \to B$ and $\Psi : A \to B$ be arbitrary quantum channels. Let $\{p_i, \rho_i\}$ and $\{q_i, \sigma_i\}$ be ensembles of states in $\mathfrak{S}(\mathcal{H}_A)$ supported by d_A-dimensional subspace $\mathcal{H}_A^0 \subseteq \mathcal{H}_A$. Then

$$|\chi(\{p_i, \Phi(\rho_i)\}) - \chi(\{q_i, \Psi(\sigma_i)\})| \leq \varepsilon \log(2d_A) + 2g(\varepsilon), \qquad (8)$$

where $\varepsilon = \frac{1}{2}\sum_i \|p_i\rho_i - q_i\sigma_i\|_1 + \beta(\Phi, \Psi)$, $g(\varepsilon) = (1+\varepsilon)h_2\left(\frac{\varepsilon}{1+\varepsilon}\right)$.

If $\Phi = \Psi$ then the factor 2 in (8) can be removed.

Continuity bound (8) is tight in the both cases $\Phi = \Psi$ and $\{p_i, \rho_i\} = \{q_i, \sigma_i\}$. The Bures distance $\beta(\Phi, \Psi)$ in (8) can be replaced by $\|\Phi - \Psi\|_\diamond^{1/2}$.

The Leung-Smith telescopic trick [CMP,292,201-215], the chain rule for $I(A\!:\!C|B)$ and lemma 2 imply the following

Proposition 3. Let $\Phi : A \rightarrow B$ and $\Psi : A \rightarrow B$ be arbitrary quantum channels, C be any system and $n \in \mathbb{N}$. Let $\rho_{A_1 \ldots A_n C}$ be a state such that $\rho_{A_1}, \ldots, \rho_{A_n}$ are finite rank states. Then

$$\left| I(B^n\!:\!C)_{\Phi^{\otimes n} \otimes \mathrm{Id}_C(\rho)} - I(B^n\!:\!C)_{\Psi^{\otimes n} \otimes \mathrm{Id}_C(\rho)} \right| \leq 2n(\varepsilon \log(2d_A) + g(\varepsilon)),$$

where $\varepsilon = \beta(\Phi, \Psi)$ and $d_A \doteq \left[\prod_{k=1}^{n} \mathrm{rank}\rho_{A_k}\right]^{1/n}$.

This continuity bound is tight (for each given n and large d_A). The Bures distance $\beta(\Phi, \Psi)$ in it can be replaced by $\|\Phi - \Psi\|_{\diamond}^{1/2}$.

If $\dim \mathcal{H}_A < +\infty$ then one can take $d_A \doteq \dim \mathcal{H}_A$.

Theorem 2. Let Φ and Ψ be quantum channels from finite-dimensional system A to arbitrary system B. Then

$$|\bar{C}(\Phi)-\bar{C}(\Psi)|\leq\varepsilon\log d_A+\varepsilon\log 2+2g(\varepsilon),\qquad(9)$$

$$|C(\Phi)-C(\Psi)|\leq 2\varepsilon\log d_A+2\varepsilon\log 2+2g(\varepsilon),\qquad(10)$$

$$|C_{\mathrm{ea}}(\Phi)-C_{\mathrm{ea}}(\Psi)|\leq 2\varepsilon\log d_A+2g(\varepsilon),\qquad(11)$$

$$|Q(\Phi)-Q(\Psi)|\leq 2\varepsilon\log d_A+2\varepsilon\log 2+2g(\varepsilon),\qquad(12)$$

$$|\bar{C}_{\mathrm{p}}(\Phi)-\bar{C}_{\mathrm{p}}(\Psi)|\leq 2\varepsilon\log d_A+2\varepsilon\log 2+4g(\varepsilon),\qquad(13)$$

$$|C_{\mathrm{p}}(\Phi)-C_{\mathrm{p}}(\Psi)|\leq 4\varepsilon\log d_A+4\varepsilon\log 2+4g(\varepsilon),\qquad(14)$$

where $d_A\doteq\dim\mathcal{H}_A$, $\varepsilon=\beta(\Phi,\Psi)$ and $g(\varepsilon)=(1+\varepsilon)h_2\left(\frac{\varepsilon}{1+\varepsilon}\right)$.

The continuity bounds (9),(11),(12) and (13) are tight.

Thank you for your attention!