Majorization preservation of Gaussian bosonic channels

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Outline

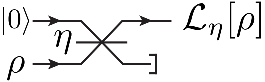
Motivation

- Gaussian quantum systems :
  easily described mathematically, easily produced in practice.
- Majorization : theory of disorder : relation with von Neumann entropy.
- Minimum output entropy conjectures : compute capacity of Gaussian channels.

Gaussian bosonic channels (1)

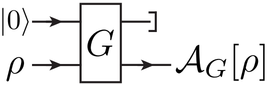
- Pure loss channel $\mathcal{L}_\eta$

$$\hat{a}_{\text{in}} \to \hat{a}_{\text{out}} = \sqrt{\eta}\hat{a}_{\text{in}} + \sqrt{1-\eta}\hat{a}_{\text{env}}$$



- Quantum-limited amplifier $\mathcal{A}_G$

$$\hat{a}_{\text{in}} \to \hat{a}_{\text{out}} = \sqrt{G}\hat{a}_{\text{in}} + \sqrt{G-1}\hat{a}_{\text{env}}$$



Gaussian bosonic channels (2)

- Adjoint $\Phi^\dagger$ of a channel Φ :

$$\mathsf{Tr}\left[A\Phi^\dagger[B]\right] = \mathsf{Tr}\left[\Phi[A]B\right], \quad \forall\ A, B.$$

  - $\mathcal{L}_\eta^\dagger = (1/\eta)\mathcal{A}_{1/\eta}$
  - $\mathcal{A}_G^\dagger = (1/G)\mathcal{L}_{1/G}$

- Any phase-insensitive Gaussian bosonic channel $\Phi = \mathcal{A}_G \circ \mathcal{L}_\eta$

Majorization (1)

$\rho \succ \sigma$ if and only if

$$\sum_{i=0}^{n} \lambda_i^{\downarrow} \geq \sum_{i=0}^{n} \mu_i^{\downarrow}, \quad \forall\, n \geq 0,$$

iff $\exists$ mixture of unitaries $\{p_n, U_n\}$ s.t.

$$\sigma = \sum_n p_n U_n \rho U_n^{\dagger}.$$

Majorization (2)

$\rho \succ \sigma$ if and only if

$$F(\rho) \geq F(\sigma), \quad \forall \text{ Schur-convex fonctions F.}$$

von Neumann entropy :

$$\rho \succ \sigma \quad \Rightarrow \quad S(\rho) \leq S(\sigma).$$

Fock-majorization

- Fock-majorization : $\rho \succ_{\mathsf{F}} \sigma$ if and only if

$$\mathsf{Tr}\left[P_n\rho\right] \geq \mathsf{Tr}\left[P_n\sigma\right], \quad \forall\ n \geq 0,$$

where $P_n = \sum_{i=0}^{n} |i\rangle\langle i|$.

Majorization and Fock-majorization

- Relation between majorization and Fock-majorization
  - $\rho^{\downarrow}$ and $\sigma^{\downarrow}$ passive : $\rho^{\downarrow} \succ_{\mathsf{F}} \sigma^{\downarrow} \quad \Leftrightarrow \quad \rho^{\downarrow} \succ \sigma^{\downarrow}$
  - $\rho^{\downarrow} \succ \sigma \quad \Rightarrow \quad \rho^{\downarrow} \succ_{\mathsf{F}} \sigma$
  - $\rho \succ_{\mathsf{F}} \sigma^{\downarrow} \quad \Rightarrow \quad \rho \succ \sigma^{\downarrow}$

- Relation between energy and Fock-majorization

$$\rho \succ_{\mathsf{F}} \sigma \quad \Rightarrow \quad \mathrm{Tr}\left[\hat{n}\rho\right] \leq \mathrm{Tr}\left[\hat{n}\sigma\right]$$

Fock-majorization in Gaussian channels

Lemma 1

Pure-loss channel $\mathcal{L}_\eta$ of arbitrary transmittance η

$$\mathcal{L}_\eta\Big[\,|k\rangle\,\langle k|\,\Big] \succ_{\mathsf{F}} \mathcal{L}_\eta\Big[\,|k+1\rangle\,\langle k+1|\,\Big], \quad \forall\ k \geq 0.$$

$$\mathcal{L}_\eta\Big[\,|k\rangle\,\langle k|\,\Big] \succ \mathcal{L}_\eta\Big[\,|k+1\rangle\,\langle k+1|\,\Big], \quad \forall\ k \geq 0.$$

Lemma 2

Quantum-limited amplifier $\mathcal{A}_G$ of arbitrary gain G

$$\mathcal{A}_G\Big[\,|k\rangle\,\langle k|\,\Big] \succ_{\mathsf{F}} \mathcal{A}_G\Big[\,|k+1\rangle\,\langle k+1|\,\Big], \quad \forall\ k \geq 0.$$

$$\mathcal{A}_G\Big[\,|k\rangle\,\langle k|\,\Big] \succ \mathcal{A}_G\Big[\,|k+1\rangle\,\langle k+1|\,\Big], \quad \forall\ k \geq 0.$$

Fock-preserving and passive-preserving channels

- Fock-preserving channel Φ :

  ρ Fock-diagonal $\Rightarrow$ $\Phi[\rho]$ Fock-diagonal.

- Fock-preserving Φ is passive-preserving :

  ρ passive $\Rightarrow$ $\Phi[\rho]$ passive.

Theorem 1

Fock-preserving Φ is passive-preserving iff

$$\Phi^\dagger\Big[\,|k\rangle\langle k|\,\Big] \succ_{\mathsf{F}} \Phi^\dagger\Big[\,|k+1\rangle\langle k+1|\,\Big], \quad \forall\, k \geq 0.$$

Corollary 1

Phase-insensitive Gaussian bosonic channels are passive preserving.

Fock-majorization preserving channels

- Fock-preserving Φ is Fock-majorization preserving :

  $\rho \succ_{\mathsf{F}} \sigma \quad \Rightarrow \quad \Phi[\rho] \succ_{\mathsf{F}} \Phi[\sigma].$

Theorem 2

Fock-preserving Φ is Fock-majorization preserving iff

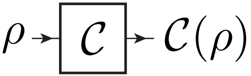
$$\Phi\Big[\,|k\rangle\,\langle k|\,\Big] \succ_{\mathsf{F}} \Phi\Big[\,|k+1\rangle\,\langle k+1|\,\Big], \quad \forall\; k \geq 0.$$

Corollary 2

Phase-insensitive Gaussian bosonic channels are Fock-majorization preserving.

Discussion (1)

Minimum output entropy conjecture : Gaussian channel $\mathcal{C}$



$$S(\rho) = S(\tau) = S_0 \qquad \Rightarrow \qquad S\Big(\mathcal{C}[\rho]\Big) \geq S\Big(\mathcal{C}[\tau]\Big)$$

Isospectral states : $\mathcal{C}[\rho^{\downarrow}] \succ \mathcal{C}[\rho]$ [1]

Comparable states : $\rho^{\downarrow} \succ \sigma \quad \Rightarrow \quad \mathcal{C}[\rho^{\downarrow}] \succ \mathcal{C}[\sigma]$

1. G. De Palma, D. Trevisan and V. Giovannetti, arXiv :1511.00293 [quant-ph]

Discussion (2)

$$\rho^{\downarrow} \succ \sigma \quad \Rightarrow \quad \mathcal{C}[\rho^{\downarrow}] \succ \mathcal{C}[\sigma]$$

If all ρ, σ comparable,

$$\begin{aligned} S\Big(\mathcal{C}[\rho^{\downarrow}]\Big) \leq \Big(\mathcal{C}[\sigma^{\downarrow}]\Big) &\Rightarrow \rho^{\downarrow} \succ \sigma^{\downarrow} \\ &\Rightarrow \rho^{\downarrow} \succ_{\mathsf{F}} \sigma^{\downarrow} \\ &\Rightarrow \mathrm{Tr}\left[\hat{n}\rho^{\downarrow}\right] \leq \mathrm{Tr}\left[\hat{n}\sigma^{\downarrow}\right] \end{aligned}$$

$$\Rightarrow \quad \tau \text{ optimal}$$